Journal of Geometry and Physics 9 (1992) 111-119
North-Holland

JOURNAL OF
GEOMETRY AND
PHYSICS

# Relative equilibria in Hamiltonian systems: The dynamic interpretation of nonlinear stability on a reduced phase space

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Received 15 November 1990
(Revised 3 March 1991)

Stability of relative equilibria for Hamiltonian systems is generally equated with Liapunov stability of the corresponding fixed point of the flow on the reduced phase space. Under mild assumptions, a sharp interpretation of this stability is given in terms of concepts on the unreduced space.

*Keywords:* *Hamiltonian systems, nonlinear stability*
*1991 MSC: 58 F 05, 58 F 10, 93 D 05*

Suppose that $(P, \omega)$ is a symplectic manifold on which a Lie group $G$ acts symplectically, and let $H : P \to \mathbb{R}$ be a $G$-invariant Hamiltonian function. A relative equilibrium is a point of phase space with Hamiltonian evolution coincident with a one parameter orbit of the symmetry group $G$. Relative equilibria correspond to fixed points of the flow on the Poisson or symplectic reduced phase spaces [1,3]. Verifying the nonlinear stability of relative equilibria is generally equated with establishing the Liapunov stability of the corresponding fixed point of the flow on the reduced phase space [3,6,9]. A defect of this approach is the absence of a fundamental interpretation of nonlinear stability in terms of the dynamics on the unreduced phase spaces.

The most obvious candidate for a definition of stability in this context is orbital stability: the evolution obtained from an initial condition near enough to a given relative equilibrium remains in any specified open neighborhood of the orbit of that relative equilibrium. In general, however, orbital stability of relative equilibria in Hamiltonian systems with symmetry cannot be expected. For example, thinking about the motion of a single rigid body rotating on its longest or shortest principal axis of inertia, then perturbing this motion in such a way that the body only rotates more quickly, one can see that two orbits in phase space result that gradually separate from one another. But notice that,

0393-0440/1992/$5.00 

after arbitrary time, the endpoints of these two orbits can be brought together by multiplying by an element in the group of rotations about that axis, and that this group is the isotropy group of the angular momentum vector.

This article proves this general statement under the conditions that the action of the isotropy group of the momentum is a proper action on the phase space, the Lie algebra of the group admits a metric invariant under the adjoint action of $G$, and no infinitesimal generator of the action of $G$ vanishes at the relative equilibrium. The situation can be described by a single sentence: *ordinarily, a stable relative equilibrium can drift only along the direction of the isotropy subgroup of its momentum.*

We begin with an appropriate definition of stability in the symmetric context. We follow the notation of ref. [1].

**Definition 1.** Let $(P, \omega, H, G, J)$ be a Hamiltonian system with symmetry and let $G'$ be a subgroup of $G$. Then a relative equilibrium $z_e$ is called *$G'$-stable*, or *stable modulo $G'$*, if for all $G'$-invariant open neighborhoods $V$ of $G' \cdot z_e$, there is an open neighborhood $U \subseteq V$ of $z_e$ which is invariant under the Hamiltonian evolution.

**Remark 2.** If $G'$ is compact, then any open neighborhood of $G' \cdot z_e$ contains a $G'$-invariant open neighborhood of $G' \cdot z_e$ (use the tube lemma of elementary topology [7]), so that in definition 1 the phrase "$G'$-invariant open neighborhoods $V$" may be replaced with "open neighborhoods $V$" in that case.

In the process of determining the stability of relative equilibria, the following easy lemma is useful.

**Lemma 3.** *Let $A$ and $B$ be bilinear forms on a finite dimensional vector space. Suppose that $A$ is positive semidefinite and that $B$ is positive definite on* $\ker A$. *Then there exists an $r > 0$ such that $A + \epsilon B$ is positive definite for all $\epsilon \in (0, r)$.*

*Proof.* Let the vector space be $E$, let $|\cdot|$ be a norm on $E$, and write $E = E' \oplus \ker A$. Then $A$ is positive definite on $E'$, so there is a constant $c_1 > 0$ such that

$$A(x_1, x_1) \geq c_1 |x_1|^2, \quad \forall x_1 \in E'.$$

Also choose $M > 0$ and $c_2 > 0$ so that

$$B(x_2, x_2) \geq c_2 |x_2|^2, \qquad \forall x_2 \in \ker A,$$
$$|B(x, y)| \leq M |x| |y|, \qquad \forall x, y \in E.$$

Then if $x_1 \in E'$ and $x_2 \in \ker A$,

$$\begin{aligned}(A + \epsilon B)(x_1 + x_2, x_1 + x_2) \\ &= A(x_1,x_1) + \epsilon B(x_1,x_1) + 2\epsilon B(x_1,x_2) + \epsilon B(x_2,x_2) \\ &\geq c_1|x_1|^2 - \epsilon M|x_1|^2 - 2\epsilon M|x_1||x_2| + \epsilon c_2|x_2|^2 \\ &= (c_1 - \epsilon M)|x_1|^2 - 2\epsilon M|x_1||x_2| + \epsilon c_2|x_2|^2.\end{aligned}$$

Viewed as a quadratic polynomial in $|x_1|$ and $|x_2|$, the discriminant of the last expression is

$$4\epsilon^2 M^2 - 4(c_1 - \epsilon M)\epsilon c_2 = -4\epsilon\big(c_1c_2 - \epsilon M(M + c_2)\big),$$

which is negative as long as $\epsilon < c_1c_2/M(M + c_2)$. □

**Remark 4.** The following alternative proof was suggested by Alan Weinstein. Use bilinearity to reduce the domains to a unit sphere in $E$. Then on a compact space $S$, if $f_1 : S \to \mathbb{R}$ is continuous and nonnegative and $f_2 : S \to \mathbb{R}$ is continuous and positive on $f_1^{-1}(0)$, then $f_1 + \epsilon f_2$ is positive for all sufficiently small positive $\epsilon$.

Points of phase space with nontrivial infinitesimal isotropy correspond to places where reduction techniques in the theory of Hamiltonian systems with symmetry fail [2]. Thus relative equilibria at these points require special attention [8]; the other relative equilibria are called *regular*. We will denote the Lie algebra of $G$ by $\mathbf{g}$.

**Definition 5.** Let $(P, \omega, H, G, J)$ be a Hamiltonian system with symmetry. A relative equilibrium $z_e$ is *regular* if, for all $\xi \in \mathbf{g}$,

$$\xi_P(z_e) = \xi \cdot z_e = \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi) \cdot z_e \neq 0.$$

A relative equilibrium that is not regular is called *degenerate*.

The main result, inspired by the energy momentum method [6], gives sufficient conditions for a regular relative equilibrium to be $G_{\mu_e}$-stable in the sense of definition 1. Here $G_{\mu_e}$ denotes the isotropy group of $\mu_e \in \mathbf{g}^*$, and $\mathbf{g}_{\mu_e}$ is the Lie algebra of $G_{\mu_e}$. Also, denote by CoAd the co-adjoint action of $G$ on $\mathbf{g}^*$.

**Theorem 6.** *Let $(P, \omega, H, G, J)$ be a Hamiltonian system with symmetry. Suppose $z_e$ is a regular relative equilibrium with evolution $t \mapsto \exp(\xi_e t) \cdot z_e$, $J(z_e) = \mu_e$, the action of $G_{\mu_e}$ on $P$ is proper, and $\mathbf{g}$ admits an inner product invariant under the adjoint action of $G_{\mu_e}$. Then $\boldsymbol{d}(H - J_{\xi_e})(z_e) = 0$, and $z_e$ is $G_{\mu_e}$-stable if it is formally stable; that is, if $\boldsymbol{d}^2(H - J_{\xi_e})(z_e) \mid T_{z_e}J^{-1}(\mu_e)$ is positive or negative definite on some (and hence any) complement to $\mathbf{g}_{\mu_e} \cdot z_e$ in $T_{z_e}J^{-1}(\mu_e)$.*

*Proof.* That $\boldsymbol{d}(H - J_{\xi_e})(z_e) = 0$ is a trivial computation using $\mathbf{X}_H(z_e) = \xi_e(z_e)$. It is also easy to see that the kernel of $\boldsymbol{d}^2(H - J_{\xi_e})(z_e) \mid T_{z_e}J^{-1}(\mu_e)$ contains $\mathbf{g}_{\mu_e} \cdot z_e$: if $\eta \in \mathbf{g}_{\mu_e}$ and $v \in T_{z_e}J^{-1}(\mu_e)$, then

$$
\begin{aligned}
\boldsymbol{d}^2(H - J_{\xi_e}(z_e))(\eta(z_e), v) &= \boldsymbol{d}(\boldsymbol{i}_\eta \boldsymbol{d}(H - J_{\xi_e}))(z_e)v \\
&= \boldsymbol{d}\{J_{\xi_e}, J_\eta\} \\
&= \boldsymbol{d}J_{[\xi_e,\eta]}(z_e)v \\
&= 0.
\end{aligned}
$$

Thus if $\boldsymbol{d}^2(H - J_{\xi_e})(z_e)$ is definite on one complement to $\mathbf{g}_{\mu_e} \cdot z_e$ in $T_{z_e}J^{-1}(\mu_e)$ then it is definite on any such complement.

Now for the proof that $z_e$ is $G_{\mu_e}$-stable. Obviously, the positive definite case may be assumed without loss of generality. The proof is obtained by modifying $H - J_{\xi_e}$, thereby constructing a $G_{\mu_e}$-invariant function $f$ in an open neighborhood of $G_{\mu_e} \cdot z_e$ which has $G_{\mu_e} \cdot z_e$ as a manifold of critical points and positive definite Hessian in directions complementary to $G_{\mu_e} \cdot z_e$. The Morse lemma is then used on a submanifold tangent to these complementary directions, and the proof is completed by establishing control on the time evolution of the function $f$.

Since the action of $G_{\mu_e}$ on $P$ is proper, it admits a relatively compact slice at $z_e$; that is, there is a submanifold $S$ containing $z_e$ with compact closure and a map $\chi$ from an open neighborhood $U_{z_e}$ of $z_e$ in $G_{\mu_e} \cdot z_e$ to $G_{\mu_e}$ such that:

- If $gz_e = z_e$ then $gS = S$.
- If $gS \cap S \neq \emptyset$ then $gz_e = z_e$.
- The map $\chi$ satisfies the following: $\chi(z_e) = \mathrm{Id}$, $\chi(u)z_e = u$ for all $u \in U_{z_e}$ and the map $U_{z_e} \times S \to P$ by $(u, z) \mapsto \chi(u) \cdot z$ is a diffeomorphism from $U_{z_e} \times S$ to some open neighborhood of $z_e$.

Indeed, $S$ may be constructed as follows: the isotropy group $I_{z_e}$ of $z_e$ in $G_{\mu_e} \cdot z_e$ is compact since the $G_{\mu_e}$ action on $P$ is proper. Thus, there is a Riemannian metric on $P$ such that the $I_{z_e}$ action is isometric. Then $S$ can be set to the image under the metric exponential map of a sufficiently small ball in the orthogonal complement of $\mathbf{g}_{\mu_e} \cdot z_e$ (the second property also requires the assumption that the action is proper.)

Note that $G_{\mu_e} \cdot S$ is an open neighborhood of $G_{\mu_e} \cdot z_e$. Construct $\pi : G_{\mu_e} \cdot S \to G_{\mu_e} \cdot z_e$ by the requirement

$$\pi(gz) = gz_e, \quad \forall z \in S, g \in G_{\mu_e}.$$

The map $\pi$ is well defined since if $gz = g'z'$ then $(g^{-1}g'S) \cap S \neq \emptyset$, so $g^{-1}g'z_e = z_e$ and hence $gz_e = g'z_e$. Also, $\pi$ is smooth since, by the definition of a slice, it is locally just a projection. Now every point in $G_{\mu_e} \cdot z_e$ is a regular relative equilibrium, so there is a smooth function $\tilde{\Psi} : G_{\mu_e} \to \mathbf{g}$ such that $\mathbf{X}_H(u) = \tilde{\Psi}(u)(u)$; that is, the evolution of $u \in G_{\mu_e} \cdot z_e$ is $t \mapsto \exp(\tilde{\Psi}(u)t)u$. It is immediate from this definition that $\tilde{\Psi}(gz) = \mathrm{Ad}_g \tilde{\Psi}(z)$ for all $g \in G_{\mu_e}$; thus the map $\Psi \stackrel{\text{def}}{=} \tilde{\Psi} \circ \pi$ has this property too, since $\pi$ intertwines the action of $G_{\mu_e}$. To summarize, we have constructed a map $\Psi : G_{\mu_e} \cdot S \to \mathbf{g}$ such that

$$\Psi(gx) = \mathrm{Ad}_g \Psi(x), \quad \forall g \in G_{\mu_e}, \tag{1}$$

and

$$\Psi(z_e) = \xi_e, \quad \mathrm{Image}\,\Psi = G_{\mu_e} \cdot \xi_e, \quad \mu_e \circ \Psi = \langle \mu_e, \xi_e \rangle. \tag{2}$$

Consider the function $f_1 = H - J_\Psi + \langle \mu_e, \xi_e \rangle - H(z_e)$. First, $f_1$ is $G_{\mu_e}$-invariant: if $g \in G_{\mu_e}$ then

$$\begin{aligned} f_1(gx) - f_1(x) &= \langle J(gx), \Psi(gx) \rangle - \langle J(x), \Psi(x) \rangle \\ &= \langle \mathrm{CoAd}_g J(x), \mathrm{Ad}_g \Psi(x) \rangle - \langle J(x), \Psi(x) \rangle \\ &= 0. \end{aligned}$$

Also, $\boldsymbol{d} f_1(z_e) = 0$: let $c(t)$ be a curve at $z_e$ tangent to $v \in T_{z_e}P$. Then

$$\begin{aligned} \boldsymbol{d} f_1(z_e)v &= \boldsymbol{d}H(z_e)v - \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \langle J(c(t)), \Psi(c(t)) \rangle \\ &= \boldsymbol{d}H(z_e)v - \langle \boldsymbol{d}J(z_e)v, \Psi(z_e) \rangle - \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \langle \mu_e, \Psi(c(t)) \rangle \\ &= [\boldsymbol{d}H(z_e)v - \boldsymbol{d}J_{\xi_e}(z_e)v] - \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \langle \mu_e, \xi_e \rangle \\ &= 0. \end{aligned}$$

Additionally, define the function $f_2 = |J - \mu_e|^2$, where the norm is obtained from the CoAd-invariant inner product induced from the hypothesized Ad-invariant inner product on $\mathbf{g}$. Obviously, $f_2$ shares with $f_1$ the properties that it is $G_{\mu_e}$-invariant and has zero derivative at $z_e$. Now let $Y$ be a complement to $T_{z_e}J^{-1}(\mu_e)$ in $T_{z_e}S$; that is, suppose

$$T_{z_e}S = (T_{z_e}S \cap T_{z_e}J^{-1}(\mu_e)) \oplus Y \stackrel{\text{def}}{=} Z \oplus Y.$$

Then $Z$ is a complement to $\mathbf{g}_{\mu_e} \cdot z_e$ in $T_{z_e}J^{-1}(\mu_e)$ and one computes that $f_1$ and $H - J_{\xi_e}$ differ be a constant on $S$, so by hypothesis $\boldsymbol{d}^2(f_1|S)(z_e)$ is positive definite on $Z$. Moreover, $\boldsymbol{d}^2(f_2|S)(z_e)$ is positive semidefinite and has kernel $Z$. Thus, by lemma 3, there is an $a \in \mathbb{R}$ such that $f = af_1 + f_2$ has $\boldsymbol{d}^2(f|S)(z_e)$ positive definite.

Thus, given a $G_{\mu_e}$-invariant neighborhood $V$ of $G_{\mu_e} \cdot z_e$, one can use the Morse lemma, and perhaps shrink $S$, to find an $\epsilon > 0$ such that $f \geq 0$ on $S$ and

$$f^{-1}[0,\epsilon) \cap S \subseteq V, \tag{3}$$

$$\mathrm{Cl}_P(f^{-1}[0,\epsilon) \cap S) \subseteq S. \tag{4}$$

Concerning the time evolution of $f$, there is the following estimate: if $F_t$ is the Hamiltonian flow, if $z \in S$, and if $F_t(z) \in G_{\mu_e} \cdot S$, then

$$\begin{aligned} & f(F_t(z)) - f(z) \\ &\quad = J_\Psi(F_t(z)) - J_\Psi(z) \\ &\quad = \langle J(F_t(z)), \Psi(F_t(z))\rangle - \langle J(z), \Psi(z)\rangle \\ &\quad = \langle J(z) - \mu_e, \Psi(F_t(z)) - \Psi(z)\rangle + \langle \mu_e, \Psi(F_t(z))\rangle - \langle \mu_e, \Psi(z)\rangle \\ &\quad = \langle J(z) - \mu_e, \Psi(F_t(z)) - \Psi(z)\rangle, \end{aligned}$$

since the evaluation of $\mu$ on the image of $\Psi$ is $\langle \mu_e, \xi_e\rangle$. Thus,

$$\begin{aligned} 0 \leq f(F_t(z)) &\leq f(z) + |\langle J(z) - \mu_e, \Psi(F_t(z)) - \Psi(z)\rangle| \\ &\leq f(z) + |J(z) - \mu_e|\,(|\Psi(F_t(z))| + |\Psi(z)|) \\ &= f(z) + 2|\xi_e||J(z) - \mu_e|. \end{aligned} \tag{5}$$

By continuity of $f$ and $J$, there is some neighborhood $S' \subseteq S$ of $z_e$ such that $|f(z)| \leq \epsilon/2$ and $|J(z) - \mu_e| \leq \epsilon/4|\xi_e|$ on $S'$. The proof will be complete if it is shown that

$$F_t(S') \subseteq f^{-1}[0,\epsilon] \cap G_{\mu_e} \cdot S \stackrel{\text{def}}{=} A, \tag{6}$$

for then $U \stackrel{\text{def}}{=} \bigcup_t F_t(G_{\mu_e} \cdot S') \subseteq A \subseteq V$, by (3) and $G_{\mu_e}$ invariance of everything in sight, and $U$ is invariant under the Hamiltonian flow. To show (6), *suppose it is false for some positive* $t$. Then for some $z \in S'$,

$$t_f \stackrel{\text{def}}{=} \sup\{\, t \mid F_s(z) \in A, \forall 0 \leq s < t \,\} < \infty.$$

Obviously, $u_f \stackrel{\text{def}}{=} F_{t_f}(z) \notin A$; otherwise, since $A$ is open, $F_t(z)$ would be contained in $A$ for a time longer than $t_f$. On the other hand, $u_f \in \mathrm{Cl}_P A$, since $t_f$ is the smallest time of escape from $A$. Thus, there are sequences $z_i \in S$

and $g_i \in G_{\mu_e}$ such that $g_i z_i \to u_f$. Since $S$ is relatively compact, one may assume $z_i \to z \in \mathrm{Cl}_P S$, and then since $G_{\mu_e}$ acts properly, some subsequence of $g_i$ converges, so one may assume $g_i \to g \in G_{\mu_e}$. Using (5), $f(z) < \epsilon$, and then using (4) gives $z \in S$. Thus, $u_f = gz \in A$, *a contradiction*. The proof that (6) is true for $t$ negative is similar. □

**Remark 7.** The conclusion that $\boldsymbol{d}(H - J_{\xi_e})(z_e) = 0$ and the definition of formal stability are, of course, not predicated on the assumption that $G_{\mu_e}$ acts properly or on the existence of an Ad-invariant inner product.

**Remark 8.** At a regular relative equilibrium, the Marsden–Weinstein reduction is well defined, at least locally, and $z_e$ passes to an equilibrium there. Formal stability is equivalent to the Hessian of the reduced Hamiltonian being positive or negative definite at that equilibrium: to see this simply use a small section to the $G_{\mu_e}$ action through $z_e$ and within $J^{-1}(\mu_e)/G_{\mu_e}$ as an open neighborhood of the equilibrium in the reduced space.

**Remark 9.** The same conclusion follows if the hypotheses are verified with the Hamiltonian $H$ replaced by any $G_{\mu_e}$-invariant conserved quantity with the same derivative as $H$ at the relative equilibrium. This is useful when dealing with degenerate relative equilibria, since isotropy implies the existence of conserved quantities with zero derivative [2] which can be used to augment the Liapunov function $H - J_\xi$ [8].

**Remark 10.** If the Lie group $G$ is compact then the action of $G_{\mu_e}$ on $P$ is proper and $\mathbf{g}$ admits an Ad-invariant metric by averaging.

**Remark 11.** It would be interesting to find conditions sufficient to guarantee $z_e$ not stable modulo any proper closed subgroup of $G_{\mu_e}$.

**Remark 12.** Suppose $z_e$ is a formally stable regular relative equilibrium with momentum $\mu_e$ and that the action of $G$ on $P$ is proper. Suppose further that $\mu_e$ is *generic*; that is, suppose that the orbits of $G_{\mu_e}$ form a local fibration of some neighborhood of $\mu_e$. Under these conditions the Poisson reduced phase space exists locally near $z_e$ and the corresponding Poisson structure has constant rank in a neighborhood of the projection of $z_e$. Then ref. [5] implies that $z_e$ is $G$-stable. Moreover, if $\mu_e$ is not generic then this conclusion is false in general (see the appendix of ref. [5].)

**Example 13.** Consider the Hamiltonian system that is two axially symmetric rigid bodies coupled by an ideal ball and socket joint moving in three dimensional space in the absence of external forces [4,8]. This system admits the symmetry group $(S^1)^2 \times SO(3)$ as well as an abundance of relative equilibria satisfying the conditions of theorem 6 [8]. Motions near those relative equilibria with nonzero total angular momentum are constrained to rotate about the axis along that vector, while motions near relative equilibria with zero total angular momentum can be expected to drift in a way not so constrained. Similar considerations apply to more general multibody systems.

**Example 14.** This example (due to Alan Weinstein) shows that the assumption of an Ad-invariant metric is essential in theorem 6. Consider two particles moving in three dimensional space with Hamiltonian function

$$H \stackrel{\text{def}}{=} \tfrac{1}{2}\|p_x\|^2 + \tfrac{1}{2}\|p_y\|^2 + \tfrac{1}{4}\|x - y\|^2.$$

Take as the Lie group of symmetries the Euclidean group: that is $SO(3) \times \mathbb{R}^3$ with the semidirect product structure

$$(A,a) \cdot (B,b) = (AB, a + Ab)$$

and acting on the configuration space $\mathbb{R}^6$ by

$$(A,a) \cdot (x,y) \stackrel{\text{def}}{=} (Ax + a, Ay + a).$$

By balancing centrifugal force with the linear attraction of the particles, the curves

$$t \mapsto \left(\exp(t\boldsymbol{j}^\wedge)\boldsymbol{k} + \gamma t, -\exp(t\boldsymbol{j}^\wedge)\boldsymbol{k} + \gamma t\right)$$

are evolutions in configuration space, where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are the usual basis vectors of $\mathbb{R}^3$ and $a^\wedge$ is the usual antisymmetric matrix generated from $a \in \mathbb{R}^3$. One verifies that these evolutions are formally stable regular relative equilibria if the vector $\gamma$ is a nonzero multiple of $\boldsymbol{j}$. Fixing one such relative equilibrium, the isotropy group of its momentum is isomorphic to $S^1 \times \mathbb{R}$, and acts on configuration space by

$$(\theta, t) \cdot (x_1; x_2) = \left(\exp(\theta\boldsymbol{j}^\wedge)x_1 + t\boldsymbol{j}, \exp(\theta\boldsymbol{j}^\wedge)x_2 + t\boldsymbol{j}\right).$$

Thus, by acting with this subgroup, points in phase space may only be translated parallel to $\boldsymbol{j}$. But nearby evolutions exist that translate along any direction, so the relative equilibrium is not stable modulo the isotropy group of the momentum. On the other hand, theorem 6 is inapplicable, since the Lie algebra of the Euclidean group does not admit a metric invariant under the adjoint action of the isotropy group of the momentum (some orbits of this action are not bounded). By ref. [5], however, the relative equilibrium is stable under the action of the full symmetry group $SO(3) \times \mathbb{R}^3$. Moreover, performing first a symplectic reduction by $\mathbb{R}^3$ (equivalent to moving to the center of mass frame), this system is reduced to a three dimensional linear oscillator, whose relative equilibria's stability can be analyzed using theorem 6.

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